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FINAL REPORT (MH-10A)

Magneto-Hydrodynamics

Contract No. AF 19(604)2138

DECEMBER, 1959

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Final Report
under
Contract No. AF 19(604)2138
Subject:

MAGNETO - HYDRODYNAMICS

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Morris Kline

Morris Kline
Project Director

1. SUMMARY

During the period beginning November 1, 1956 and ending December 31, 1959 the magneto-hydrodynamics group of the Division of Electromagnetics, Institute of Mathematical Sciences at New York University has carried on research in hydromagnetic theory under Contract No. AF 19(604)2138. In accordance with the contract, the objective has been to conduct studies and mathematical investigations of the motion of charged particles of an ionized fluid in the presence of external electric and magnetic fields, with particular applications to motions in the ionized layers of the atmosphere and to the influences such motions have on the geomagnetic field at or near the earth's surface.

In Section 2 of this report Prof. J. Bazer and Dr. R. Lewis discuss work in progress at the time of the contract's termination. Abstracts of all completed work are given in Section 3. A list of reports and publications is appended (Section 4). Section 1.1 summarizes the completed and published work of Prof. J. Bazer, Mr. W B Ericson and Mr. O. Fleischman, and Section 2.2, that of Dr. R. Kraichnan.

The work of the following research assistants and associates was supported by contract funds: A. Brown, Drs. H. Hochstadt, R. Lewis, B. Zumino, R. Kraichnan, J. Bazer and S. Borowitz. Contract funds also supported the graduate work of O. Fleischman and W.B Ericson, who received his Ph.D. for work on hydromagnetic shocks during the period

1.1 Non-Linear and Linearized Hydromagnetic Flow - J. Bazer, W.B. Ericson and O. Fleischman

It was observed several years ago by K.O. Friedrichs [1] that the hydromagnetic equations for a perfect, infinitely conducting fluid share with the equations of compressible fluid dynamics the property of being a first order 'symmetric hyperbolic' system of 'conservation laws'. This observation meant that the equations of magnetohydrodynamics could be treated by methods similar to those employed in ordinary gas dynamics. Specifically, it meant, at least in principle, that i) one could develop a theory of one-dimensional* hydromagnetic simple waves which together with hydromagnetic shocks could be used to solve initial value problems with sufficiently simple conditions; ii) one could develop a theory of geometrical hydromagnetics dealing with the propagation of small disturbances, similar to the existing theories of geometrical acoustics and optics; and finally iii) one could develop a theory of two-dimensional** steady hydromagnetic flow by employing suitable generalizations of methods used in the steady flow theory of gas dynamics. These three observations constituted the basis of a program initiated by us from the outset. Parts i) and ii) have essentially been carried out. The following is a survey of the problems treated and some of the results obtained.

In the report cited above, K.O. Friedrichs developed a simple wave theory of one-dimensional hydromagnetic motion and discussed briefly the motion that results when a shear-flow discontinuity is present initially. A detailed description of this motion was given by Bazer in [2]***. This discussion applies equally well to a flow which is initially parallel to a perfectly conducting rigid wall. The basis of both problems is that a shear-flow discon-

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[*] By one-dimensional hydromagnetic flow we mean a hydromagnetic flow in which the dependent variables, the magnetic intensity, fluid velocity, density, etc., are functions of one space variable and the time t; no restrictions are placed on the orientation of the magnetic and flow vectors.

[**] Let $\vec{B}$, $\vec{u}$, ρ be the magnetic induction, particle velocity and mass density, respectively. In two dimensional steady flow $\vec{B}$, $\vec{u}$ and ρ depend on x and y alone and $\vec{B}$ and $\vec{u}$ are confined to the (x,y)-plane.

[***] Numbers in square brackets refer to references on p.14.

tinuity is an 'inadmissible' hydromagnetic contact discontinuity if a component of the magnetic field - B_n say - normal to the surface of discontinuity is present. It turns out, if $B_n \neq 0$ and a shear flow discontinuity is present initially, that a motion in the direction of the normal $\vec{n}$ will result and that this motion can be completely described in terms of hydromagnetic simple waves and shocks. Two results are of special interest. The first is that cavitation will result if the initial shear flow discontinuity is large enough. The second is that a transverse (transverse to $\vec{n}$) magnetic field is created even when such a component is absent initially and that this field may be made arbitrarily large simply by making the initial shear flow large enough.

Next, in [3], J. Bazer and W.B. Ericson turned to the problem of determining and of classifying, solely by analytical means, all admissible solutions (shocks and contact discontinuities) of the hydromagnetic discontinuity relations. The state in front of - i.e., on the low density side of - the shock was assumed to be known; no restriction was placed on the orientation of the magnetic field in front. This problem had been treated earlier by De Hoffmann and Teller [4], Helfer [5], Lust [6], and Friedrichs [7]. It was shown in [3] that, apart from certain limit shocks (e.g., gas shocks), the shock velocity and the quantities characterizing the state behind hydromagnetic shocks could be expressed as simple algebraic functions of the jump in the transverse magnetic field across the shock. This fact enabled them to achieve a natural classification of all hydromagnetic shocks and to derive several new properties of these shocks.

In [7] J. Bazer and O.W. Fleischman treated the propagation of weak hydromagnetic discontinuities. As in geometrical optics, it was shown that the wave fronts evolving from a given initial manifold - there will be six of them in general - could be constructed by means of rays. In addition, formulas were derived giving the variation of the discontinuity 'strength' of a given propagating mode along rays associated with the mode. These results were then employed to obtain an explicit solution of a simple mixed initial-boundary-value problem which was designed to illustrate a method for producing hydromagnetic disturbances and to exhibit the fact that a given initial disturbance gives rise, in general, to several propagating modes.

1.2 Research in Statistical Hydromagnetics - R.H. Kraichnan

In the study of turbulence without magnetic fields, useful theories have been constructed from essentially dimensional arguments, a notable success being the Kolmogorov similarity theory. Similar attempts in the analysis of hydromagnetic turbulence meet with much more severe difficulties because the greater number of physical parameters which are of importance do not appear to permit unambiguous dimensional results in most cases of interest.

This situation provides a strong motivation for the development of an analytical theory of turbulent motion which proceeds deductively from the equations of motion rather than from qualitative considerations. Reports MH-6, MH-7 and MH-9 describe parts of a program with this objective. Revised and substantially improved versions of these reports have since appeared as journal publications [10]. The abstracts of the published versions of these papers are given in Section 3. The first paper is devoted to a critical examination of previous attempts at a deductive theory of turbulence based on the assumption of a normal relation between second-order and fourth-order moments of the velocity field. The second and third papers describe aspects of a new theory based on the systematic use of a novel perturbation treatment of the equations of motion.

2. WORK IN PROGRESS

2.1 Reflection and Refraction of Weak Discontinuities - J. Bazer

A weak, slow Alfvén or fast hydromagnetic disturbance wave impinges on a (in general) curved surface of discontinuity Γ. Various types of Γ's are considered with a view to later applications - - e.g., Γ may be a contact-discontinuity surface, a rigid infinitely conducting surface, etc. The problem is to determine the hydromagnetic analogs of Snell's laws and, for each Γ, to calculate the reflection and transmission coefficients associated with a given incident mode. Thus far, we have succeeded in determining the HM-Snell's laws and have analyzed them graphically and in part numerically. The reflection coefficients associated with each of the three possible types of exciting modes have been determined when Γ is a rigid infinitely conducting surface and the magnetic field is in the plane of incidence. In general, we have found that a given incident mode gives rise to three reflected and (when transmission is possible) to three refracted waves (Alfvén slow and fast).

2.2 Motion of Charged Sheets Moving in an Electromagnetic Field - J. Bazer

The general problem of moving charged sheets in electromagnetic fields may be described as follows: Let $S(t)$ be a surface in free space which consists of an electrically charged fluid and which moves under the influence of an electromagnetic field. Temperature and pressure effects within the surface layer are neglected. For the sake of concreteness the fluid is assumed to be composed of identical particles of mass m and charge e. Our formalism, however, may easily be extended to the case of n sheets $S_j(t)$,

$j = 1,2,\ldots,n$, composed of particles of mass m_j and charge e_j (some or all of the sheets may coincide) and, in addition, to the case of a continuous fluid. Initially the shape and position of the surface and the surface charge density are assumed given. The problem is to determine i) the motion of the surface and the fluid particles which comprise it, and ii) the charge density and the electromagnetic field at all subsequent times.

The jump conditions across moving charged sheets and the proper form of the equations of motion have been determined. In addition we have succeeded in obtaining the electromagnetic and dynamical equations of motion from a single Hamilton's principle. The virtue of employing this principle, apart from the fact that one may take into account the symmetries of the field from the outset, is the following. In some instances one may carry out the spatial integrations in the Lagrangian density before performing the variations and thereby reduce the problem of determining the motion of the system to a problem involving the motion of circuits in which the inductances and capacitances vary with time.

2.3 Plane Waves in a Plasma - R. Lewis

The behavior of a plasma consisting of electrons, positive ions and neutral particles, in the presence of a uniform external magnetic field, is described in terms of Maxwell's equations and a system of linearized Boltzmann equations with simplified collision term. For plane wave propagation, the equations can be solved exactly. This leads to an explicit expression for a conductivity tensor which reduces to the conductivity tensor of the Magneto-ionic theory in an appropriate asymptotic limit. Using the calculated expression for the conductivity, the dispersion relation can be solved in various asymptotic limits. For zero collision frequency these solutions reduce to the earlier results of Landau, Gross, Gordeyev, Bernstein, etc.

Status: A report is being prepared.

2.4 A Solution of the Equations of Statistical Mechanics - R. Lewis

The solution of the initial value problem for Bogoliubov's functional differential equation of non-equilibrium statistical mechanics is obtained. This equation is equivalent to the infinite system of 'hierarchy equations'. The solution is then expanded in an infinite power series in the density which has the advantage that the calculation of the leading terms requires the solution of s-body problems only for small values of s.

It is hoped that the above work may be generalized to include systems of charged particles.

A report on the work relating to neutral particles has been issued under Contract No. AF 49(638)341, Report No. HT-3.

Status: We are now working on the corresponding problem for charged particles.

3. COMPLETED WORK

3.1 MH-5: Resolution of an Initial Shear-Flow Discontinuity in One-Dimensional Hydromagnetic Flow - J. Bazer (Abstract)

The problem of the resolution of an initial shear-flow discontinuity in an infinitely conducting, compressible fluid is treated. The discussion applies equally well to a flow which is initially parallel to a perfectly conducting wall. The basis of both problems is that a shear flow is an 'inadmissible' hydromagnetic contact discontinuity if a component of the magnetic field normal to the surface of discontinuity is present. The motion which results when a shear-flow discontinuity is present initially is investigated. It is shown that this motion may be described in terms of known solutions of the hydromagnetic shock and continuum equations. The dependence of the final constant state of the medium on the initial state is analyzed by analytical and numerical means. Formulae for various quantities of interest are given for the limiting cases of weak and strong initial shear flows.

3.2 MH-6 Relation of Fourth-Order to Second-Order Moments in Stationary Isotropic Turbulence - Robert H. Kraichnan (Abstract)

An investigation is made of the hypothesis that the fourth-order moments of the two-time velocity amplitude distribution in stationary, isotropic, incompressible turbulence are related to second-order moments as in a normal distribution. It is concluded that this hypothesis is inconsistent with the equations of motion, and the inconsistency is exhibited as a gross violation of energy conservation in the inertial range. The origin of the inconsistency is discussed. The arguments developed are used to demonstrate inconsistencies in Chandrasekhar's recent theory of turbulence. Qualitative considerations are presented with regard to the consistency of the hypothesis that fourth-order moments of the simultaneous amplitude distribution are related to second-order moments as in a normal distribution. The general validity of this restricted hypothesis is questioned also.

3.3 MH-7 Irreversible Statistical Mechanics of Incompressible Hydromagnetic Turbulence - Robert H. Kraichnan (Abstract)

The irreversible statistical mechanics of incompressible hydromagnetic turbulence driven by external forces is treated by methods which do not require that the system be close to a state of detailed balance. The equations of motion are expressed in terms of linearly independent modes formed from the wave-vector components of velocity and magnetic fields, and the nonlinear interaction is exhibited as the sum of individually conservative three-mode interactions. A fundamental statistical equation is constructed giving necessary and sufficient conditions for all members of a distribution of time-functions to satisfy the equations of motion; it involves only second-, third-, and fourth-order distribution moments. A variational criterion is proposed for specifying a distribution consistent with the fundamental equation under physically appropriate constraints. It leads to a complete formal solution of the statistical problem. This solution is not exploited. Instead, two statistical hypotheses based on the assumption of high mode density are advanced. With their aid, each three-mode interaction is treated as a small perturbation on the motion due to all the three-mode interactions and the external forces. The moments in the fundamental equation for the stationary case thereby are expressed in terms of the diagonal elements of the time-covariance matrix and distribution-averaged infinitesimal-impulse-response matrix of the system. Closed equations are obtained which fix these matrix elements in terms of the covariance matrix of the external forces. If the statistical hypotheses are sound, this provides a theory of unbounded turbulence (infinite mode density) driven by Gaussian-distributed homogeneous forces which is exact at all Reynolds numbers based on rms velocity and the macroscale determined by the driving forces. The general theory is specialized to obtain integro-differential equations determining the covariance scalars and modal impulse-response functions for stationary, isotropic hydromagnetic turbulence. In the nonmagnetic case, the asymptotic inertial-range solution yields the wave number spectrum $E(k) = 2\pi c(\epsilon v_o)^{\frac{1}{2}}k^{-\frac{3}{2}}$ and the modal time-autocorrelation function $J_1(2v_o k\tau)/(v_o k\tau)$, where v_o is the rms velocity in any direction, ϵ is the mean rate of energy-cascade/unit-mass, and c is a universal number fixed by the theory. This contradicts the Kolmogorov similarity hypotheses; independent arguments are advanced against the latter.

3.4 Hydromagnetic Shocks - J. Bazer and W.B. Ericson (Abstract)

All physically admissible solutions of the hydromagnetic shock relations for an ideal polytropic gas are exhibited and classified by purely analytical means. The state ahead of the shock (i.e., on the low-density side) is assumed to be known; no restriction is placed on the direction of the magnetic field in front. It is shown, apart from certain 'limit' shocks (e.g., pure gas shocks), that the shock velocity and the quantities characterizing the state behind hydromagnetic shocks may be expressed as simple algebraic functions of the discontinuity in the magnetic field across the shock. A natural classification of all hydromagnetic shocks, based on this representation of the state behind the shock, is given. Several useful analytical properties of the various types of hydromagnetic shocks are derived. The results are illustrated graphically for the case of an ideal monatomic gas. The relation between earlier schemes of classification and the present scheme is discussed.

3.5 The Structure of Isotropic Turbulence at Very High Reynolds Numbers - Robert H. Kraichnan (Abstract)

A recapitulation is first given of a recent theory of homogeneous turbulence based on the condition that the Fourier amplitudes of the velocity field be as randomly distributed as the dynamical equations permit. This theory involves the average infinitesimal-impulse-response functions of the Fourier amplitudes and employs a new kind of perturbation method which yields what are believed to be exact expansions of third- and higher-order statistical moments of the Fourier amplitudes in terms of second-order moments and these response functions.

In the present paper the theory is applied in lowest approximation (called the direct-interaction approximation) to stationary isotropic turbulence of very high Reynolds numbers. The characteristic wave-number $k_o = \epsilon/v_o^3$ and Reynolds number $R_o = v_o k_o^{-1}/\nu$, where v_o is the r.m.s. velocity in any given direction, ϵ is the power dissipated per unit mass, and ν is the kinematic viscosity, are introduced. For $R_o^{\frac{1}{3}} \gg 1$, it is found that the inertial and dissipation ranges extend over wave-numbers k satisfying $k_o \ll k \ll R_o k_o$. The time-correlation and average infinitesimal-impulse-response functions of the Fourier amplitudes in these ranges are evaluated. They are found to be asymptotically identical and given by $J_1(2v_o k\tau)/v_o k\tau$, where τ is the time interval.

The energy spectrum in these ranges is determined by a non-linear integral equation, involving the time-correlation and response functions, which is suitable for solution by iteration. The solution is of the form $E(k)/v_o\nu = (k/k_d)^{\frac{3}{2}} f(k/k_d)$, where $E(k)$ is the three-dimensional spectrum function, $k_d = R_o^{\frac{2}{3}} k_o$ is a wave-number characterizing the dissipation range, and $f(k/k_d)$ is a universal function. In the inertial range, $E(k) = f(0)(\epsilon v_o)^{\frac{1}{2}} k^{-\frac{3}{2}}$, asymptotically. The parameter $f(0)$ can be obtained by quadratures, without solving the integral equation for $E(k)$. Spectral energy transport throughout the inertial and dissipation ranges is found to proceed by a cascade process essentially local in wave-number space; the direct power delivered by all modes below k to all modes above $k' \gg k$ is of order $\epsilon(k/k')^{\frac{3}{2}}$ if k and k' both lie within the inertial range. The mean-square velocity derivatives of all orders are bound to be finite. For $R_o^{\frac{1}{3}} \gg 1$ the skewness factor of the distribution of the n^{th}-order longitudinal velocity derivative is found to have the asymptotic form $A_n R_o^{-\frac{1}{6}}$, where A_n is a universal constant.

The theory is compared with experiment and is found to be slightly better supported than the Kolmogorov theory. However, it is stressed that extreme caution must be exercised in interpreting the experimental evidence as support for either theory.

An analysis is given of the relations between the Kolmogorov theory, Heisenberg's heuristic theory, the analytical theories of Heisenberg and Chandrasekhar, the theories of Proudman & Reid and Tatsumi, and the present theory.

3.6 Propagation of Weak Hydromagnetic Discontinuities - J. Bazer and O. Fleishman (Abstract)

A theoretical discussion is given of the propagation of weak hydromagnetic discontinuities (e.g., weak shocks) in an infinitely conducting, perfect, compressible fluid. The undisturbed flow is assumed to be steady and isentropic. As in geometrical optics, the wave fronts evolving from a given initial manifold - there will be six of them, in general - are constructed by means of rays. In addition, formulas are derived describing the variation of the discontinuity 'strength' of a given propagating mode along rays associated with that mode. These results are employed to obtain an explicit solution of a simple mixed initial, boundary-value problem that has been designed to illustrate, (1) a method of producing hydromagnetic disturbances, and (2) the fact that an initial disturbance gives rise, in general, to several propagating waves.

4. RESEARCH REPORTS SUBMITTED AND PUBLICATIONS

Under Contract No. AF. 19(604)-926

MH-1	J.B. Keller	Electrohydrodynamics. I. The Equilibrium of a Charged Gas in a Container Published - J. Rat. Mech., 5, 715 (1956). Math. Rev., 18, 442 (1957).	Jan. '55
MH-2	B.A. Lippman	The Decay of Magnetic Fields	May '55
MH-3	B.A. Lippman	The Decay of a Stellar Magnetic Field	June '55
MH-4	O. Fleischman and B.A. Lippman	Remarks on Alfvén's Perturbation Method	Aug. '55

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MH-5	J. Bazer	Resolution of an Initial Shear Flow Discontinuity in One-Dimensional Hydromagnetic Flow Published - Astrophys.J.,128,686(1958).	June '56
MH-6	R.H. Kraichnan	Relation of Fourth to Second Moments in Stationary Homogeneous Hydromagnetic Turbulence Published - Phys.Rev.,107,1485 (1957).	Jan. '57
MH-7	R.H. Kraichnan	Statistical Mechanics of Stationary Homogeneous Hydromagnetic Turbulence Published - Phys.Rev.109,1407 (1958).	Apr. '59
MH-8	W.B. Ericson and J. Bazer	Hydromagnetic Shocks Published - Astrophys.J.,129,758(1959).	Jan. '58
MH-9	R.H. Kraichnan	The Structure of Isotropic Turbulence at Very High Reynolds Numbers Published - J.Fluid Mech.,5,497 (1959).	Feb. '58
MH-10	J. Bazer and O. Fleischmann	Propagation of Weak Hydromagnetic Discontinuities Published - Physics of Fluids,2,366(1959).	May '58

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- (b) Propagation of electromagnetic waves, New York University (1949).

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Irreversible statistical mechanics of incompressible hydromagnetic turbulends; Phys. Rev., 109, 1407 (1958).

The structure of isotropic turbulence at very high Reynolds numbers;J. of Fluid Mech.,5, 497 (1959

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